On the Propagation of Waves through a Stratified Medium, with Special Reference to the Question of Reflection.

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The medium is supposed to be such that its properties are everywhere a function of but one co-ordinate x, being of one uniform quality where x is less than a certain value x_1, and of another uniform quality (in general, different from the first) where x exceeds a greater value x_{m-1}; and the principal problem is the investigation of the reflection which in general ensues when plane waves in the first medium are incident upon the stratifications. For the present we suppose the quality to be uniform through strata of finite thickness, the first transition occurring when $x = x_1$, the second at $x = x_2$, and the last at $x = x_{m-1}$.

The expressions for the waves in the various media in order may be taken to be

$$\phi_1 = \mathrm{A}_1 e^{i[ct+by-a_1(x-x_1)]} + \mathrm{B}_1 e^{i[ct+by+a_1(x-x_1)]}, \qquad (1)$$

$$\phi_2 = \mathrm{A}_2 e^{i[ct+by-a_2(x-x_1)]} + \mathrm{B}_2 e^{i[ct+by+a_2(x-x_1)]},$$

$$\phi_3 = \mathrm{A}_3 e^{i[ct+by-a_3(x-x_2)]} + \mathrm{B}_3 e^{i[ct+by+a_3(x-x_2)]},$$

and so on, the A's and B's denoting arbitrary constants. The first terms represent the waves travelling in the positive direction, the second those travelling in the negative direction; and our principal aim is the determination of the ratio $\mathrm{B}_1/\mathrm{A}_1$ imposed by the conditions of the problem, including the requirement that in the final medium there shall be no negative wave.

As in the simple transition from one uniform medium to another ('Theory of Sound,' § 270), the symbols c and b are common to all the media, the first depending merely upon the periodicity, while the constancy of the second is required in order that the traces of the various waves on the surfaces of transition should move together—equivalent to the ordinary law of refraction. In the usual optical notation, if V be the velocity of propagation and θ the angle of incidence,

$$c = 2\pi\mathrm{V}/\lambda, \qquad b = (2\pi/\lambda)\sin\theta, \qquad a = (2\pi/\lambda)\cos\theta, \qquad (2)$$

where V/λ, $\lambda^{-1}\sin\theta$ are the same in all the strata. On the other hand a is variable and is connected with the direction of propagation within the stratum by the relation

$$a = b\cot\theta. \qquad (3)$$

The a's are thus known in terms of the original angle of incidence and of the various refractive indices.

Since the factor $e^{i(ct+by)}$ runs through all our expressions, we may regard it as understood and write simply

$$\phi_1 = \mathrm{A}_1 e^{-ia_1(x-x_1)} + \mathrm{B}_1 e^{ia_1(x-x_1)}, \tag{4}$$

$$\phi_2 = \mathrm{A}_2 e^{-ia_2(x-x_1)} + \mathrm{B}_2 e^{ia_2(x-x_1)}, \tag{5}$$

$$\phi_3 = \mathrm{A}_3 e^{-ia_3(x-x_2)} + \mathrm{B}_3 e^{ia_3(x-x_2)}, \tag{6}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi_m = \mathrm{A}_m e^{-ia_m(x-x_{m-1})} + \mathrm{B}_m e^{ia_m(x-x_{m-1})}. \tag{7}$$

In the problem of reflection we are to make $\mathrm{B}_m = 0$, and (if we please) $\mathrm{A}_m = 1$.

We have now to consider the boundary conditions which hold at the surfaces of transition. In the case of sound travelling through gas, where ϕ is taken to represent the velocity-potential, these conditions are the continuity of $d\phi/dx$ and of $\sigma\phi$, where σ is the density. Whether the multiplier attaches to the dependent variable itself or to its derivative is of no particular significance. For example, if we take a new dependent variable ψ, equal to $\sigma\phi$, the above conditions are equivalent to the continuity of ψ and of $\sigma^{-1}d\psi/dx$. Nor should we really gain generality by introducing a multiplier in both places. We may therefore for the present confine ourselves to the acoustical form, knowing that the results will admit of interpretation in numerous other cases.

At the first transition $x = x_1$ the boundary conditions give

$$a_1(\mathrm{B}_1 - \mathrm{A}_1) = a_2(\mathrm{B}_2 - \mathrm{A}_2), \qquad \sigma_1(\mathrm{B}_1 + \mathrm{A}_1) = \sigma_2(\mathrm{B}_2 + \mathrm{A}_2). \tag{8}$$

If we stop here, we have the simple case of the juxtaposition of two media both of infinite depth. Supposing $\mathrm{B}_2 = 0$, we get

$$\frac{\mathrm{B}_1}{\mathrm{A}_1} = \frac{\sigma_2/\sigma_1 - a_2/a_1}{\sigma_2/\sigma_1 + a_2/a_1} = \frac{\sigma_2/\sigma_1 - \cot\theta_2/\cot\theta_1}{\sigma_2/\sigma_1 + \cot\theta_2/\cot\theta_1}. \tag{9}$$

For a further discussion of (9) reference may be made to 'Theory of Sound' (*loc. cit.*). In the case of the simple gases the compressibilities are the same, and $\sigma_1 \sin^2\theta_1 = \sigma_2 \sin^2\theta_2$. The general formula (9) then identifies itself with Fresnel's expression

$$\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}. \tag{10}$$

On the other hand, if $\sigma_2 = \sigma_1$, the change being one of compressibility only, we find

$$(9) = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)}, \tag{11}$$

Fresnel's other expression.

In the above it is supposed that a_2 (and θ_2) are real. If the wave be incident in the more refractive medium and the angle of incidence be too great, a_2 becomes imaginary, say $-ia_2'$. In this case, of course, the reflection is total, the modulus of (9) becoming unity. The change of phase incurred is given by (9). In accordance with what has been said these results are at once available for the corresponding optical problems.

If there are more than two media, the boundary conditions at $x = x_2$ are—

$$a_2\{B_2e^{ia_2(x_2-x_1)}-A_2e^{-ia_2(x_2-x_1)}\} = a_3(B_3-A_3), \tag{12}$$

$$\sigma_2\{B_2e^{ia_2(x_2-x_1)}+A_2e^{-ia_2(x_2-x_1)}\} = \sigma_3(B_3+A_3), \tag{13}$$

and so on. For extended calculations it is desirable to write these equations in an abbreviated shape. We set

$$B_2-A_2 = H_2, \qquad B_2+A_2 = K_2, \quad \text{etc.}, \tag{14}$$

$$\cos a_2(x_2-x_1) = c_1, \qquad i\sin a_2(x_2-x_1) = s_1, \quad \text{etc.} \tag{15}$$

$$a_3/a_2 = \alpha_2, \qquad \sigma_3/\sigma_2 = \beta_2, \quad \text{etc.}; \tag{16}$$

and the series of equations then takes the form

$$H_1 = \alpha_1 H_2, \qquad K_1 = \beta_1 K_2, \tag{17}$$

$$c_1H_2+s_1K_2 = \alpha_2H_3, \qquad s_1H_2+c_1K_2 = \beta_2K_3, \tag{18}$$

$$c_2H_3+s_2K_3 = \alpha_3H_4, \qquad s_2H_3+c_2K_3 = \beta_3K_4, \tag{19}$$

and so on. In the reflection problem the special condition is the numerical equality of H and K of highest suffix. We may make

$$H = -1, \qquad K = +1. \tag{20}$$

As we have to work backwards from the terms of highest suffix, it is convenient to solve algebraically each pair of simple equations. In this way, remembering that $c^2-s^2 = 1$, we get

$$H_1 = \alpha_1H_2, \qquad K_1 = \beta_1K_2, \tag{21}$$

$$H_2 = c_1\alpha_2H_3-s_1\beta_2K_3, \qquad K_2 = -s_1\alpha_2H_3+c_1\beta_2K_3, \tag{22}$$

$$H_3 = c_2\alpha_3H_4-s_2\beta_3K_4, \qquad K_3 = -s_2\alpha_3H_4+c_2\beta_3K_4, \tag{23}$$

and so on. In these equations the c's and the β's are real, and also the α's, unless there is "total reflection"; the s's are pure imaginaries, with the same reservation.

When there are three media, we are to suppose in the problem of reflection that $H_3 = -1$, $K_3 = 1$. Thus from (21), (22),

$$H_1 = -\alpha_1(c_1\alpha_2+s_1\beta_2), \qquad K_1 = \beta_1(s_1\alpha_2+c_1\beta_2);$$

so that

$$\frac{B_1}{A_1} = \frac{K_1+H_1}{K_1-H_1} = \frac{c_1(\beta_1\beta_2-\alpha_1\alpha_2)+s_1(\alpha_2\beta_1-\alpha_1\beta_2)}{c_1(\beta_1\beta_2+\alpha_1\alpha_2)+s_1(\alpha_2\beta_1+\alpha_1\beta_2)}. \tag{24}$$

If there be no "total reflection," the relative intensity of the reflected waves is

$$\frac{c_1^2(\beta_1\beta_2-\alpha_1\alpha_2)^2-s_1^2(\alpha_2\beta_1-\alpha_1\beta_2)^2}{c_1^2(\beta_1\beta_2+\alpha_1\alpha_2)^2-s_1^2(\alpha_2\beta_1+\alpha_1\beta_2)^2}, \tag{25}$$

where $$c_1^2=\cos^2 a_2(x_2-x_1), \qquad -s_1^2=\sin^2 a_2(x_2-x_1). \tag{26}$$

The reflection will vanish independently of the values of c_1 and s_1, *i.e.*, whatever may be the thickness of the middle layer, provided

$$\beta_1\beta_2-\alpha_1\alpha_2=0, \quad \alpha_2\beta_1-\alpha_1\beta_2=0; \quad \text{or} \quad \beta_1=\alpha_1, \quad \beta_2=\alpha_2,$$

since these quantities are all positive. Reference to (9) shows that these are the conditions of vanishing reflection at the two surfaces of transition considered separately.

If these conditions be not satisfied, the evanescence of (25) requires that either c_1 or s_1 be zero. The latter case is realised if the intermediate layer be abolished, and the remaining condition is equivalent to $\sigma_3/\sigma_1=a_3/a_1$, as was to be expected from (9). We learn now that, if there would be no reflection in the absence of an intermediate layer, its introduction will have no effect provided $a_2(x_2-x_1)$ be a multiple of π. An obvious example is when the first and third media are similar, as in the usual theory of "thin plates."

On the other hand, if c_1, or $\cos a_2(x_2-x_1)$, vanish, the remaining requirement for the evanescence of (25) is that $\beta_2/\alpha_2=\beta_1/\alpha_1$.

In this case $$\frac{\beta_1-\alpha_1}{\beta_1+\alpha_1}=\frac{\beta_2-\alpha_2}{\beta_2+\alpha_2};$$

so that by (9) the reflections at the two faces are equal in all respects.

In general, if the third and first media are similar, (25) reduces to

$$\frac{\{\beta_1/\alpha_1-\alpha_1/\beta_1\}^2\sin^2 a_2(x_2-x_1)}{4\cos^2 a_2(x_2-x_1)+\{\beta_1/\alpha_1+\alpha_1/\beta_1\}^2\sin^2 a_2(x_2-x_1)}, \tag{27}$$

which may readily be identified with the expression usually given in terms of (9).

It remains to consider the cases of so-called total reflection. If this occurs only at the *second* surface of transition, a_1, a_2 are real, while a_3 is a pure imaginary. Thus α_1 is real, and α_2 is imaginary; c_1 is real always, and s_1 is imaginary as before; the β's are always real. Thus, if we separate the real and imaginary parts of the numerator and denominator of (24), we get

$$\frac{\mathrm{B}_1}{\mathrm{A}_1}=\frac{c_1\beta_1\beta_2+s_1\alpha_2\beta_1-c_1\alpha_1\alpha_2-s_1\alpha_1\beta_2}{c_1\beta_1\beta_2+s_1\alpha_2\beta_1+c_1\alpha_1\alpha_2+s_1\alpha_1\beta_2}, \tag{28}$$

of which the modulus is unity. In this case, accordingly, the reflection back in the first medium is literally total, whatever may be the thickness of the intermediate layer, as was to be expected.

The separation of real and imaginary parts follows the same rule when a_2 is imaginary, as well as a_3. For then α_1 is imaginary, while α_2, s_1 are real. Thus $s_1\alpha_2\beta_1$ remains real, and $c_1\alpha_1\alpha_2$, $s_1\alpha_1\beta_2$ remain imaginary. The reflection back in the first medium is total in this case also.

The only other case requiring consideration occurs when a_2 is imaginary and a_3 real. The reflection is then total if the middle layer be thick enough, but if this thickness be reduced, the reflection cannot remain total, as is evident if we suppose the thickness to vanish. The ratios α_1, α_2 are now both imaginary, while s_1 is real. The separation of real and imaginary parts stands as in (24), and the intensity of reflection is still expressed by (25). If we take $a_2 = -ia_2'$, we may write in place of (25),

$$\frac{(\beta_1\beta_2-\alpha_1\alpha_2)^2\cosh^2 a_2'(x_2-x_1)-(\alpha_2\beta_1-\alpha_1\beta_2)^2\sinh^2 a_2'(x_2-x_1)}{(\beta_1\beta_2+\alpha_1\alpha_2)^2\cosh^2 a_2'(x_2-x_1)-(\alpha_2\beta_1+\alpha_1\beta_2)^2\sinh^2 a_2'(x_2-x_1)}. \qquad (29)$$

When x_2-x_1 is extremely small, this reduces to

$$\frac{(\beta_1\beta_2-\alpha_1\alpha_2)^2}{(\beta_1\beta_2+\alpha_1\alpha_2)^2}, \quad \text{or} \quad \frac{(\sigma_3/\sigma_1-a_3/a_1)^2}{(\sigma_3/\sigma_1+a_3/a_1)^2},$$

in accordance with (9).

When on the other hand x_2-x_1 exceeds a few wave-lengths, (29) approaches unity, as we see from a form, equivalent to (29), viz.,

$$\frac{(\beta_1^2-\alpha_1^2)(\beta_2^2-\alpha_2^2)\cosh^2 a_2'(x_2-x_1)+(\alpha_2\beta_1-\alpha_1\beta_2)^2}{(\beta_1^2-\alpha_1^2)(\beta_2^2-\alpha_2^2)\cosh^2 a_2'(x_2-x_1)+(\alpha_2\beta_1+\alpha_1\beta_2)^2}. \qquad (30)$$

It is to be remembered that in (30), α_1^2, α_2^2, $\alpha_1\alpha_2$ have negative values.

The form assumed when the third medium is similar to the first may be noted. In this case $\alpha_1\alpha_2 = 1$, $\beta_1\beta_2 = 1$, and we get from (29)

$$\frac{(\beta_1/\alpha_1-\alpha_1/\beta_1)^2\sinh^2 a_2'(x_2-x_1)}{(\beta_1/\alpha_1-\alpha_1/\beta_1)^2\sinh^2 a_2'(x_2-x_1)-4}. \qquad (31)$$

In this case, of course, the reflection vanishes when x_2-x_1 is sufficiently reduced.

Equations (21), etc., may be regarded as constituting the solution of the general problem. If there are m media, we suppose $H_m = -1$, $K_m = 1$, thence calculate in order from the pairs of simple equations H_{m-1}, K_{m-1}; H_{m-2}, K_{m-2}, etc., until H_1 and K_1 are reached; and then determine the ratio B_1/A_1. The procedure would entail no difficulty in any special case numerically given; but the algebraic expression of H_1 and K_1 in terms of H_m and K_m soon becomes complicated, unless further simplifying conditions are introduced. Such simplification may be of two kinds. In the first it is supposed that the total thickness between the initial and final media is small relatively to the wave-lengths, so that the phase-changes occurring within the layer are of subordinate importance. In the second kind of simplification the thicknesses are left arbitrary, but the changes in the character of the medium, which occur at each transition, are supposed small.

The problem of a thin transitional layer has been treated by several authors, L. Lorenz,* Van Ryn,† Drude,‡ Schott,§ and Maclaurin.|| A full account will be found in 'Theory of Light' by the last named. It will therefore not be necessary to treat the subject in detail here; but it may be worth while to indicate how the results may be derived from our equations. For this purpose it is convenient to revert to the original notation so far as to retain a and σ. Thus in place of (17), etc., we write

$$a_1 H_1 = a_2 H_2, \qquad \sigma_1 K_1 = \sigma_2 K_2. \tag{32}$$

$$a_2(c_1 H_2 + s_1 K_2) = a_3 H_3, \qquad \sigma_2(s_1 H_2 + c_1 K_2) = \sigma_3 K_3, \quad \text{etc.} \tag{33}$$

In virtue of the supposition that all the layers are thin, the c's are nearly equal to unity and the s's are small. Thus, for a first approximation we identify c with 1 and neglect s altogether, so obtaining

$$a_1 H_1 = a_2 H_2 = \ldots = a_m H_m, \qquad \sigma_1 K_1 = \sigma_2 K_2 = \ldots = \sigma_m K_m. \tag{34}$$

The relation of H_1, K_1 to H_m, K_m is the same as if the transition between the extreme values took place without intermediate layers, and the law of reflection is not disturbed by the presence of these layers, as was to be expected.

For the second approximation we may still identify the c's with unity, while the s's are retained as quantities of the first order. Adding together the column of equations constituting the first members of (32), (33), etc., we find

$$a_1 H_1 + a_2 s_1 K_2 + a_3 s_2 K_3 + \ldots + a_{m-1} s_{m-2} K_{m-1} = a_m H_m; \tag{35}$$

and in like manner, with substitution of σ for a and interchange of K and H,

$$\sigma_1 K_1 + \sigma_2 s_1 H_2 + \ldots + \sigma_{m-1} s_{m-2} H_{m-1} = \sigma_m K_m. \tag{36}$$

In the small terms containing s's we may substitute the approximate values of H and K from (34). For the problem of reflection we suppose $H_m + K_m = 0$. Hence

$$\frac{H_1}{K_1} = -\frac{\sigma_1 a_m}{\sigma_m a_1} \frac{1 + \frac{\sigma_m}{a_m} \sum \frac{a_2 s_1}{\sigma_2}}{1 + \frac{a_m}{\sigma_m} \sum \frac{\sigma_2 s_1}{a_2}}. \tag{37}$$

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* 'Pogg. Ann.,' 1860, vol. 111, p. 460.

† 'Wied. Ann.,' 1883, vol. 20, p. 22.

‡ 'Wied. Ann.,' 1891, vol. 43, p. 126.

§ 'Phil. Trans.,' 1894, vol. 185, p. 823.

|| 'Roy. Soc. Proc.,' A, 1905, vol. 76, p. 49.

In (37) $s_1 = ia_2(x_2-x_1)$, and so on, so that

$$\Sigma\frac{a_2 s_1}{\sigma_1} = i\int\frac{a^2 dx}{\sigma}, \qquad \Sigma\frac{\sigma_2 s_1}{a_2} = i\int\sigma\,dx, \tag{38}$$

the integration extending over the layer of transition.

One conclusion may be drawn at once. To this degree of approximation the reflection is independent of the order of the strata. It will be noted that the sums in (37) are pure imaginaries. In what follows we shall suppose that a_m is real.

As the final result for the reflection, we find

$$\frac{B_1}{A_1} = \frac{K_1+H_1}{K_1-H_1} = Re^{i\delta}, \tag{39}$$

where

$$R = \frac{\sigma_m/\sigma_1 - a_m/a_1}{\sigma_m/\sigma_1 + a_m/a_1}, \tag{40}$$

$$\tan\delta = 2\,\frac{\dfrac{a_m}{\sigma_m}\displaystyle\int\sigma\,dx - \dfrac{\sigma_m}{a_m}\displaystyle\int\frac{a^2 dx}{\sigma}}{\dfrac{a_1\sigma_m}{a_m\sigma_1} - \dfrac{a_m\sigma_1}{a_1\sigma_m}}. \tag{41}$$

To this order of approximation the *intensity* of the reflection is unchanged by the presence of the intermediate layers, unless, indeed, the circumstances are such that (40) is itself small. If $\sigma_m/\sigma_1 = a_m/a_1$ absolutely, we have

$$R = \tfrac{1}{2}\left\{\frac{a_m}{\sigma_m}\int\sigma\,dx - \frac{\sigma_m}{a_m}\int\frac{a^2 dx}{\sigma}\right\} \tag{42}$$

and $\delta = \frac{1}{2}\pi$. This case is important in Optics, as representing the reflection at the polarising angle from a contaminated surface.

The two important optical cases: (i) where σ is constant, leading (when there is no transitional layer) to Fresnel's formula (11), and (ii) where $\sigma\sin^2\theta$ is constant, leading to (10), are now easily tretaed as special examples. Introducing the refractive index μ, we find after reduction for case (i)

$$\delta' = -\frac{4\pi\cos\theta_1}{\lambda_1(\mu_m^2-\mu_1^2)}\int(\mu_m^2-\mu^2)\,dx, \tag{43}$$

where λ_1, μ_1, relate to the first medium, μ_m is the index for the last medium, and the integration is over the layer of transition. The application of (43) should be noticed when the layer is in effect abolished, either by supposing $\mu = \mu_m$, or, on the other hand, $\mu = \mu_1$.

In the second case (42), corresponding to the polarising angle, becomes

$$R = \frac{\pi}{\lambda_1\mu_1\sqrt{(\mu_m^2+\mu_1^2)}}\int\frac{(\mu^2-\mu_1^2)(\mu^2-\mu_m^2)}{\mu^2}\,dx. \tag{44}$$

In general for this case

$$\delta'' = -\frac{4\pi\cos\theta_1}{\lambda_1}\,\frac{\cos^2\theta_1\int(\mu_m^2-\mu^2)\,dx-\sin^2\theta_1\int(\mu_m^2-\mu^2)\left(\frac{\mu_1^2}{\mu^2}+\frac{\mu_1^2}{\mu_m^2}-1\right)dx}{(\mu_m^2-\mu_1^2)\left(\cos^2\theta_1-\frac{\mu_1^2}{\mu_m^2}\sin^2\theta_1\right)}. \tag{45}$$

The second fraction in (45) is equal to the thickness of the layer of transition simply, when we suppose $\mu = \mu_1$.

Further, $$\delta''-\delta' = -\frac{4\pi}{\lambda_1}\,\frac{\cos\theta_1\sin^2\theta_1}{\mu_m^2-\mu_1^2}\,\frac{\int\frac{(\mu_m^2-\mu^2)(\mu^2-\mu_1^2)}{\mu}\,dx}{\cos^2\theta_1-\frac{\mu_1^2}{\mu_m^2}\sin^2\theta_1}, \tag{46}$$

the difference of phase vanishing, as it ought to do, when $\mu = \mu_1$, or μ_m, or, again, when $\theta_1 = 0$.

It should not escape notice that the expressions (10) and (11) have different signs when θ_1 and θ_2 are small. This anomaly, as it must appear from an optical point of view, should be corrected when we consider the significance of $\delta''-\delta'$. The origin of it lies in the circumstance that, in our application of the boundary conditions, we have, in effect, used different vectors as dependent variables to express light of the two polarisations. For further explanation reference may be made to former writings, *e.g.* "On the Dynamical Theory of Gratings."*

If throughout the range of integration, μ is intermediate between the terminal values μ_1, μ_m, the reflection is of the kind called positive by Jamin. The transition may well be of this character when there is no contamination. On the other hand, the reflection is negative, if μ has throughout a value outside the range between μ_1 and μ_m. It is probable that something of this kind occurs when water has a greasy surface.

The formulæ required in Optics, viz. (43), (44), (45), (46), are due, in substance, to Lorenz and van Ryn. The more general expressions (41), (42), do not seem to have been given.

There is no particular difficulty in pursuing the approximation from (32), etc. At the next stage the second term in the expansion of the *c*'s must be retained, while the *s*'s are still sufficiently represented by the first terms. The result, analogous to (37), (38), is

$$\frac{H_1}{K_1} = -\frac{\sigma_1 a_m}{\sigma_m a_1}\,\frac{1-\int_0^d\sigma\,.\int_0^x\frac{a^2\,dx}{\sigma}\,.\,dx+i\frac{\sigma_m}{a_m}\int_0^d\frac{a^2}{\sigma}\,dx}{1-\int_0^d\frac{a^2}{\sigma}\,.\int_0^x\sigma\,dx\,.\,dx+i\frac{a_m}{\sigma_m}\int_0^d\sigma\,dx}, \tag{47}$$

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* 'Roy. Soc. Proc.,' A, 1907, vol. 79, p. 413.

in which the terminal abscissæ of the variable layer are taken to be 0 and d, instead of x_1 and x_{m-1}. I do not follow out the application to particular cases such as $\sigma =$ constant, or $\sigma \sin^2\theta =$ constant. For this reference may be made to Maclaurin, who, however, uses a different method.

The second case which allows of a simple approximate expression for the reflection arises when all the partial reflections are small. It is then hardly necessary to appeal to the general equations: the method usually employed in optics suffices. The assumptions are that at each surface of transition the incident waves may be taken to be the same as in the first medium, merely retarded by the appropriate amount, and that each partial reflection reaches the first medium no otherwise modified than by such retardation. This amounts to the neglect of waves three times reflected. Thus

$$\frac{\mathrm{B}_1}{\mathrm{A}_1} = \frac{\beta_1 - \alpha_1}{\beta_1 + \alpha_1} + \frac{\beta_2 - \alpha_2}{\beta_2 + \alpha_2} e^{-2ia_2(x_2 - x_1)} + \frac{\beta_3 - \alpha_3}{\beta_3 + \alpha_3} e^{-2i[a_2(x_2 - x_1) + a_3(x_3 - x_2)]} + \ldots . \quad (48)$$

An interesting question suggests itself as to the manner in which the transition from one uniform medium to another must be effected in order to obviate reflection, and especially as to the least thickness of the layer of transition consistent with this result. If there be two transitions only, the least thickness of the layer is obtained by supposing in (48)

$$\frac{\beta_1 - \alpha_1}{\beta_1 + \alpha_1} = \frac{\beta_2 - \alpha_2}{\beta_2 + \alpha_2} \quad (49)$$

and

$$2a_2(x_2 - x_1) = \pi; \quad (50)$$

and this conclusion, as we have seen already, is not limited to the case of small differences of quality. In its application to perpendicular incidence, (50) expresses that the thickness of the layer is one-quarter of the wave-length proper to the layer. The two partial reflections are equal in magnitude and sign. It is evident that nothing better than this can be done so long as the reflections are of one sign, however numerous the surfaces of transition may be.

If we allow the partial reflections to be of different signs, some reduction of the necessary thickness is possible. For example, suppose that there are two intermediate layers of equal thickness, of which the first is similar to the final uniform medium, and the second similar to the initial uniform medium. Of the three partial reflections the first and third are similar, but the second is of the opposite sign. If three vectors of equal numerical value compensate one another, they must be at angles of 120°. The necessary conditions are satisfied (in the case of perpendicular transmission) if the total thickness ($2l$) is $\frac{1}{6}\lambda$, in accordance with

$$1 - e^{-4\pi i l/\lambda} + e^{-8\pi i l/\lambda} = 0.$$

The total thickness of the layer of transition is thus somewhat reduced, but only by a very artificial arrangement, such as would not usually be contemplated when a layer of transition is spoken of. If the progress from the first to the second uniform quality be always *in one direction*, reflection cannot be obviated unless the layer be at least $\frac{1}{4}\lambda$ thick.

The general formula (48) may be adapted to express the result appropriate to continuous variation of the medium. Suppose, for example, that σ is constant, making $\beta = 1$, and corresponding to the continuity of both ϕ and $d\phi/dx$.[*] It is convenient to suppose that the variation commences at $x = 0$. Then (48) may be written

$$\frac{B_1}{A_1} = -\int \frac{da}{2a} e^{-2i\int_0^x a\,dx}, \tag{51}$$

a at any point x being connected with the angle of propagation by the usual relation (3). In the special case of perpendicular propagation, $a = 2\pi\mu/\lambda_1\mu_1$, μ being refractive index and λ_1, μ_1 relating to the first medium.

A curious example, theoretically possible even if unrealisable in experiment, arises when the variable medium is constituted in such a manner that the velocity of propagation is everywhere constant, so that there is no refraction. Then a is constant, $\alpha = 1$, and (48) gives

$$\frac{B_1}{A_1} = \int \frac{d\sigma}{2\sigma} e^{-2iax}. \tag{52}$$

Some of the questions relating to the propagation of waves in a variable medium are more readily treated on the basis of the appropriate differential equation. As in (1), we suppose that the waves are plane, and that the medium is stratified in plane strata perpendicular to x, and we usually omit the exponential factors involving t and y, which may be supposed to run through. In the case of perpendicular propagation, y would not appear at all.

Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \kappa^2\phi = 0, \tag{53}$$

in which (unless κ^2 can be infinite) it is necessary to suppose that ϕ and $d\phi/dx$ are continuous; κ^2 is a function of x, which must be everywhere positive when the transmission is perpendicular, as, for example, in the case of a stretched string. When the transmission is oblique to the strata,

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[*] These would be the conditions appropriate to a stretched string of variable longitudinal density vibrating transversely.

κ^2 may become negative, corresponding to "total reflection," but in most of what follows we shall assume that this does not happen. The continuity of ϕ and $d\phi/dx$, even though κ^2 be discontinuous, appears to limit the application of (53) to certain kinds of waves, although, as a matter of analysis, the general differential equation of the second order may always be reduced to this form.*

In the theory of a uniform medium, we may consider stationary waves or progressive waves. The former may be either

$$\phi = \mathrm{A}\cos\kappa_0 x \cos pt, \quad \text{or} \quad \phi = \mathrm{B}\sin\kappa_0 x \sin pt;$$

and, if $\mathrm{B} = \pm\mathrm{A}$, the two may be combined, so as to constitute progressive waves

$$\phi = \mathrm{A}\cos(pt\pm\kappa_0 x).$$

Conversely, progressive waves, travelling in opposite directions, may be combined so as to constitute stationary waves. When we pass to variable media, no ambiguity arises respecting stationary waves; they are such that the phase is the same at all points. But is there such a thing as a progressive wave? In the full sense of the phrase there is not. In general, if we contemplate the wave forms at two different times, the difference between them cannot be represented by a mere shift of position proportional to the interval of time which has elapsed.

The solution of (53) may be taken to be

$$\phi = \mathrm{A}'\psi(x)+\mathrm{B}'\chi(x), \tag{54}$$

where $\psi(x)$, $\chi(x)$, are real oscillatory functions of x; A', B', arbitrary constants as regards x. If we introduce the time-factor, writing p in place of the less familiar c of (1), we may take

$$\phi = \mathrm{A}\cos pt\,.\,\psi(x)+\mathrm{B}\sin pt\,.\,\chi(x); \tag{55}$$

and this may be put into the form

$$\phi = \mathrm{H}\cos(pt-\theta), \tag{56}$$

where $$\mathrm{H}\cos\theta = \mathrm{A}\psi(x), \qquad \mathrm{H}\sin\theta = \mathrm{B}\chi(x), \tag{57}$$

or $$\mathrm{H}^2 = \mathrm{A}^2[\psi(x)]^2+\mathrm{B}^2[\chi(x)]^2, \tag{58}$$

$$\theta = \tan^{-1}\frac{\mathrm{B}\,.\,\chi(x)}{\mathrm{A}\,.\,\psi(x)}. \tag{59}$$

But the expression for ϕ in (56) cannot be said to represent in general a progressive wave. We may illustrate this even from the case of the uniform medium where $\psi(x) = \cos\kappa x$, $\chi(x) = \sin\kappa x$. In this case (56) becomes

$$\phi = \{\mathrm{A}^2\cos^2\kappa x+\mathrm{B}^2\sin^2\kappa x\}^{\frac{1}{2}}\cos\left\{pt-\tan^{-1}\left(\frac{\mathrm{B}}{\mathrm{A}}\tan\kappa x\right)\right\}. \tag{60}$$

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* Forsyth's 'Differential Equations,' § 59.

If $B = \pm A$, reduction ensues to the familiar positive or negative progressive wave. But if B be not equal to $\pm A$, (55), taking the form

$$\phi = \tfrac{1}{2}(A+B)\cos(pt-\kappa x)+\tfrac{1}{2}(A-B)\cos(pt+\kappa x),$$

clearly does not represent a progressive wave. The mere possibility of reduction to the form (57) proves little, without an examination of the character of H and θ.

It may be of interest to consider for a moment the character of θ in (60). If B/A, or, say, m, is positive, θ may be identified with κx at the quadrants, but elsewhere they differ, unless $m = 1$. Introducing the imaginary expressions for tangents, we find

$$\theta = \kappa x+M\sin 2\kappa x+\tfrac{1}{2}M^2\sin 4\kappa x+\tfrac{1}{3}M^3\sin 6\kappa x+\dots, \tag{61}$$

where
$$M = \frac{m-1}{m+1}. \tag{62}$$

When κ is constant, one of the solutions of (53) makes ϕ proportional to $e^{-i\kappa x}$. Acting on this suggestion, and following out optical ideas, let us assume in general

$$\phi = \eta e^{-i\int a\,dx}, \tag{63}$$

where the amplitude η and a are real functions of x, which, for the purpose of approximations, may be supposed to vary slowly. Substituting in (53), we find

$$\frac{d^2\eta}{dx^2}+(\kappa^2-a^2)\,\eta-2ia^{\frac{1}{2}}\frac{d}{dx}(a^{\frac{1}{2}}\eta) = 0. \tag{64}$$

For a first approximation, we neglect $d^2\eta/dx^2$. Hence

$$\kappa = a, \qquad \kappa^{\frac{1}{2}}\eta = C, \tag{65}$$

so that
$$\phi = C\kappa^{-\frac{1}{2}}e^{ipt}e^{-i\int\kappa\,dx}, \tag{66}$$

or in real form,
$$\phi = C\kappa^{-\frac{1}{2}}\cos\left(pt-\int\kappa\,dx\right). \tag{67}$$

If we hold rigorously to the suppositions expressed in (65), the satisfaction of (64) requires that $d^2\eta/dx^2 = 0$, or $d^2\kappa^{-\frac{1}{2}}/dx^2 = 0$. With omission of arbitrary constants affecting merely the origin and the scale of x, this makes $\kappa^2 = x^{-4}$, corresponding to the differential equation

$$x^4\frac{d^2\phi}{dx^2}+\phi = 0, \tag{68}$$

whose accurate solution is accordingly

$$\phi = Cxe^{i(pt-1/x)}. \tag{69}$$

In (69) the imaginary part may be rejected. The solution (69) is, of course, easily verified. In all other cases (67) is only an approximation.

As an example, the case where $\kappa^2 = n^2/x^2$ may be referred to. Here $\int \kappa dx = n \log x - \epsilon$, and (67) gives

$$\phi = \mathrm{C}x^{\frac{1}{2}} \cos(pt - n \log x + \epsilon) \tag{70}$$

as an approximate solution. We shall see presently that a slight change makes it accurate.

Reverting to (64), we recognise that the first and second terms are real, while the third is imaginary. The satisfaction of the equation requires therefore that

$$a^{\frac{1}{2}}\eta = \mathrm{C}, \tag{71}$$

and that
$$\kappa^2 = \mathrm{C}^4\eta^{-4} - \frac{1}{\eta}\frac{d^2\eta}{dx^2}; \tag{72}$$

while (63) becomes
$$\phi = \eta e^{-i\mathrm{C}^2\int \eta^{-2}dx}. \tag{73}$$

Let us examine in what cases η may take the form $\mathrm{D}x^r$. From (72),

$$\kappa^2 = \mathrm{C}^4\mathrm{D}^{-4}x^{-4r} - r(r-1)x^{-2}. \tag{74}$$

If $r = 0$, κ^2 is constant. If $r = 1$, $\kappa^2 = \mathrm{C}^4\mathrm{D}^{-4}x^{-4}$, already considered in (68). The only other case in which κ^2 is a simple power of x occurs when $r = \frac{1}{2}$, making

$$\kappa^2 = (\mathrm{C}^4\mathrm{D}^{-4} + \tfrac{1}{4})x^{-2} = n^2x^{-2} \text{ (say)}. \tag{75}$$

Here $\eta = \mathrm{D}x^{\frac{1}{2}}$, $\mathrm{C}^2\int \eta^{-2}dx = \mathrm{C}^2/\mathrm{D}^2 . \log x - \epsilon$, and the realised form of (73) is

$$\phi = \mathrm{D}x^{\frac{1}{2}} \cos\{pt - \sqrt{(n^2 - \tfrac{1}{4})} \log x + \epsilon\}, \tag{76}$$

which is the exact form of the solution obtained by approximate methods in (70). For a discussion of (76) reference may be made to 'Theory of Sound,' second edition, § 148*b*.

The relation between a and η in (71) is the expression of the energy condition, as appears readily if we consider the application to waves along a stretched string. From (53), with restoration of e^{ipt},

$$\frac{d\phi}{dt} = e^{ipt}e^{-i\int a dx} . ip\eta, \qquad \frac{d\phi}{dx} = e^{ipt}e^{-i\int a dx}\left\{\frac{d\eta}{dx} - ia\eta\right\}.$$

If the common phase factors be omitted, the parts of $d\phi/dt$ and $d\phi/dx$ which are in the same phase are as $p\eta$ and $a\eta$, and thus the mean work transmitted at any place is as $a\eta^2$. Since there is no accumulation of energy between two places, $a\eta^2$ must be constant.

When the changes are gradual enough, a may be identified with κ, and then $\eta \propto \kappa^{-\frac{1}{2}}$, as represented in (67).

If we regard η as a given function of x, a follows when C has been chosen, and also κ^2 from (72). In the case of perpendicular propagation κ^2 cannot be negative, but this is the only restriction. When η is constant, κ^2 is constant; and thus if we suppose η to pass from one constant value to another through a finite transitional layer, the transition is also from one uniform κ^2 to another; and (73) shows that there is no reflection back into the first medium. If the terminal values of η and therefore of κ^2 be given, and the transitional layer be thick enough, it will always be possible, and that in an infinite number of ways, to avoid a negative κ^2, and thus to secure complete transmission without reflection back; but if with given terminal values the layer be too much reduced, κ^2 must become negative. In this case reflection cannot be obviated.

It may appear at first sight as if this argument proved too much, and that there should be no reflection in any case so long as κ^2 is positive throughout. But although a constant η requires a constant κ^2, it does not follow conversely that a constant κ^2 requires a constant η, and, in fact, this is not true. *One* solution of (72), when κ^2 is constant, certainly is $\eta^2 = \mathrm{C}^2/\kappa$; but the complete solution necessarily includes two arbitrary constants, of which C is not one. From (60) it may be anticipated that a solution of (72) may be

$$\eta^2 = \mathrm{A}^2\cos^2\kappa x + \mathrm{B}^2\sin^2\kappa x = \tfrac{1}{2}(\mathrm{A}^2+\mathrm{B}^2) + \tfrac{1}{2}(\mathrm{A}^2-\mathrm{B}^2)\cos 2\kappa x. \qquad (77)$$

From this we find on differentiation

$$\eta^3\frac{d^2\eta}{dx^2} + \kappa^2\eta^4 = \kappa^2\mathrm{A}^2\mathrm{B}^2;$$

and thus (72) is satisfied, provided that

$$\kappa^2\mathrm{A}^2\mathrm{B}^2 = \mathrm{C}^2. \qquad (78)$$

It appears then that (77) subject to (78) is a solution of (72). The second arbitrary constant evidently takes the form of an arbitrary addition to x, and η will not be constant unless $\mathrm{A}^2 = \mathrm{B}^2$.

On the supposition that η and a are slowly varying functions, the approximations of (65) may be pursued. We find

$$\eta = \mathrm{C}\kappa^{-\frac{1}{2}}\left\{1 - \tfrac{1}{4}\kappa^{-\frac{3}{2}}\frac{d^2\kappa^{-\frac{1}{2}}}{dx^2}\right\}, \qquad (79)$$

$$a = \kappa + \tfrac{1}{2}\kappa^{-\frac{1}{2}}\frac{d^2\kappa^{-\frac{1}{2}}}{dx^2}. \qquad (80)$$

The retardation, as usually reckoned in optics, is $\int\kappa dx$. The additional retardation according to (80) is

$$\tfrac{1}{2}\int\kappa^{-\frac{1}{2}}\frac{d^2\kappa^{-\frac{1}{2}}}{dx^2}dx = \tfrac{1}{2}\left[\kappa^{-\frac{1}{2}}\frac{d\kappa^{-\frac{1}{2}}}{dx}\right] - \tfrac{1}{2}\int\left(\frac{d\kappa^{-\frac{1}{2}}}{dx}\right)^2 dx.$$

As applied to the transition from one uniform medium to another, the retardation is *less* than according to the first approximation by

$$\tfrac{1}{2}\int\left(\frac{d\kappa^{-\frac{1}{2}}}{dx}\right)^2 dx. \qquad (81)$$

The supposition that η varies slowly excludes more than a very small reflection.

Equations (79), (80) may be tested on the particular case already referred to where $\kappa = n/x$. We get

$$\eta = \mathrm{C}n^{-\frac{1}{2}}x^{\frac{1}{2}}\left(1+\frac{1}{16n^2}\right), \qquad a = \frac{1}{x}\left(n-\frac{1}{8n}\right);$$

so that

$$\int a\,dx = \left(n-\frac{1}{8n}\right)\log x.$$

When n^{-4} is neglected in comparison with unity, $n-\frac{1}{8}n^{-1}$ may be identified with $\sqrt{(n^2-\frac{1}{4})}$.

Let us now consider what are the possibilities of avoiding reflection when the transition layer (x_2-x_1) between two uniform media is reduced. If η_1, κ_1; η_2, κ_2 are the terminal values, (72) requires that

$$\kappa_1^2 = \mathrm{C}^4\eta_1^{-4}, \qquad \kappa_2^2 = \mathrm{C}^4\eta_2^{-4}.$$

We will suppose that $\eta_2 > \eta_1$. If the transition from η_1 to η_2 be made too quickly, viz., in too short a space, $d^2\eta/dx^2$ will become somewhere so large as to render κ^2 negative. The same consideration shows that at the beginning of the layer of transition (x_1), $d\eta/dx$ must vanish. The quickest admissible rise of η will ensue when the curve of rise is such as to make κ^2 vanish. When η attains the second prescribed value η_2, it must suddenly become constant, notwithstanding that this makes κ^2 positively infinite.

From (72) it appears that the curve of rise thus defined satisfies

$$\frac{d^2\eta}{dx^2} = \mathrm{C}^4\eta^{-3}. \tag{82}$$

The solution of (82), subject to the conditions that $\eta = \eta_1$, $d\eta/dx = 0$, when $x = x_1$, is

$$\eta^2-\eta_1^2 = \mathrm{C}^4\eta_1^{-2}(x-x_1)^2 = \kappa_1^2\eta_1^2(x-x_1)^2. \tag{83}$$

Again, when $\eta = \eta_2$, $x = x_2$, so that

$$\kappa_1^2(x_2-x_1)^2 = \frac{\eta_2^2-\eta_1^2}{\eta_1^2} = \frac{\kappa_1-\kappa_2}{\kappa_2}, \tag{84}$$

giving the minimum thickness of the layer of transition.

It will be observed that the minimum thickness of the layer of transition necessary to avoid reflection diminishes without limit with $\kappa_1-\kappa_2$, that is, as the difference between the two media diminishes. However, the arrangement under discussion is very artificial. In the case of the string, for example, it is supposed that the density drops suddenly from the first uniform value to zero, at which it remains constant for a time. At the end of this it becomes momentarily infinite, before assuming the second uniform value. The infinite longitudinal density at x_2 is equivalent to a finite load there attached. In the layer of transition, if so it may be called, the string remains straight during the passage of the waves.

If, as in the more ordinary use of the term, we require the transition to be such that κ^2 moves always in one direction from the first terminal value to the second, the problem is one already considered. The minimum thickness is such that κ^2 has throughout it a constant intermediate value, so chosen as to make the reflections equal at the two faces.

It would be of interest to consider a particular case in which κ^2 varies continuously and always in the one direction. As appears at once from (72), $d^2\eta/dx^2$, as well as $d\eta/dx$, must vanish at both ends of the layer, and there must also be a third point of inflection between. If the layer be from $x = 0$ to $x = \beta$, we may take

$$\frac{d^2\eta}{dx^2} = \mathrm{A}x(x-\alpha)(x-\beta). \tag{85}$$

We find that $\beta = 2\alpha$, and that

$$\frac{\eta-\eta_1}{\eta_2-\eta_1} = \frac{15x^3}{4\alpha^5}\left\{\frac{x^2}{20}-\frac{\alpha x}{4}+\frac{\alpha^2}{3}\right\}, \tag{86}$$

$$\frac{1}{\eta_2-\eta_1}\frac{d^2\eta}{dx^2} = \frac{15x}{4\alpha^5}(x-\alpha)(x-2\alpha). \tag{87}$$

From these κ^2 would have to be calculated by means of (72), and one question would be to find how far α might be reduced without interfering with the prescribed character of κ^2. But to discuss this in detail would lead us too far.

If the differences of quality in the variable medium are small, (72) simplifies. If η_0, κ_0 be corresponding values, subject to $\kappa_0^2 = \mathrm{C}^4\eta_0^{-4}$, we may take

$$\eta = \eta_0+\eta', \qquad \kappa^2 = \kappa_0^2+\delta\kappa^2, \tag{88}$$

where η' and $\delta\kappa^2$ are small, and (72) becomes approximately

$$\frac{d^2\eta'}{dx^2}+4\kappa_0^2\eta' = -\eta_0\delta\kappa^2. \tag{89}$$

Replacing x by t, representing *time*, we see that the problem is the same as that of a pendulum upon which displacing forces act; see 'Theory of Sound,' § 66. The analogue of the transition from one uniform medium to another is that of the pendulum initially at rest in the position of equilibrium, upon which at a certain time a displacing force acts. The force may be variable at first, but ultimately assumes a constant value. If there is to be no reflection in the original problem, the force must be of such a character that when it becomes constant the pendulum is left *at rest* in the new position. If the object be to effect the transition between the two states in the shortest possible time, but with forces which are restricted never to exceed the final value, it is pretty evident that the force must immediately assume the maximum admissible value, and retain it for such a time that the pendulum, then left free, will just reach the new position of equilibrium, after which the force is reimposed. The present solution is excluded, if it be required that the force never decrease in value. Under this restriction the best we can do is to make the force assume at once *half* its final value, and remain constant for a time equal to one-half of the free period. Under this force the pendulum will just swing out to the new position of equilibrium, where it is held on arrival by doubling the force. These cases have already been considered, but the analogue of the pendulum is instructive.

Kelvin* has shown that the equation of the second order

$$\frac{d}{dx}\left(\frac{1}{\mathrm{P}}\frac{dy_1}{dx}\right) = y_1 \tag{90}$$

can be solved by a machine. It is worth noting that an equation of the form (53) is solved at the same time. In fact, if we make

$$y_1 = \frac{dy_2}{dx}, \qquad \mathrm{P}y_2 = \frac{dy_1}{dx}, \tag{91}$$

we get on elimination either (90) for y_1, or

$$\frac{d^2y_2}{dx^2} = \mathrm{P}y_2 \tag{92}$$

for y_2. Equations (91) are those which express directly the action of the machine.

It now remains to consider more in detail some cases where total reflection occurs. When there is merely a simple transition from one medium (1) to another (2), the transmitted wave is

$$\phi_2 = \mathrm{A}_2 e^{-ia_2(x-x_1)} e^{i(ct+by)}. \tag{93}$$

If there is total reflection, a_2 becomes imaginary, say $-ia_2'$; the transmitted wave is then no longer a wave in the ordinary sense, but there remains some disturbance, not conveying energy, and rapidly diminishing as we recede from the surface of transition according to the factor $e^{-a_2'(x-x_1)}$. From (2)

$$a_2^2 = \frac{4\pi^2}{\lambda_2^2}\cos^2\theta_2 = \frac{4\pi^2}{\lambda_1^2}\left(\frac{\mathrm{V}_1^2}{\mathrm{V}_2^2} - \sin^2\theta_1\right),$$

or

$$a_2' = \frac{2\pi}{\lambda_1}\sqrt{(\sin^2\theta_1 - \mathrm{V}_1^2/\mathrm{V}_2^2)}. \tag{94}$$

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* 'Roy. Soc. Proc.,' 1876, vol. 24, p. 269.

It appears that soon after the critical angle is passed, the disturbance in the second medium extends sensibly to a distance of only a few wave-lengths.

The circumstances of total reflection at a sudden transition are thus very simple; but total reflection itself does not require a sudden transition, and takes place however gradual the passage may be from the first medium to the second, the only condition being that when the second is reached the angle of refraction becomes imaginary. From this point of view total reflection is more naturally regarded as a sort of refraction, reflection proper depending on some degree of abruptness of transition. Phenomena of this kind are familiar in Optics under the name of *mirage.*

In the province of acoustics the vagaries of fog-signals are naturally referred to irregular refraction and reflection in the atmosphere, due to temperature or wind differences; but the difficulty of verifying a suggested explanation on these lines is usually serious, owing to our ignorance of the state of affairs overhead.*

The penetration of vibrations into a medium where no regular waves can be propagated is a matter of considerable interest; but, so far as I am aware, there is no discussion of such a case, beyond that already sketched, relating to a sudden transition between two uniform media. It might have been supposed that oblique propagation through a variable medium would involve too many difficulties, but we have already had opportunity to see that, in reality, obliquity need not add appreciably to the complication of the problem.

To fix ideas, let us suppose that we are dealing with waves in a membrane uniformly stretched with tension T, and of superficial density ρ, which is a function of x only. The equation of vibration is ('Theory of Sound,' § 194)

$$\mathrm{T}\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right) = \rho\frac{d^2\phi}{dt^2},$$

or, if ϕ be proportional to $e^{i(ct+by)}$, as in (1),

$$\frac{d^2\phi}{dx^2}+(c^2\rho/\mathrm{T}-b^2)\,\phi = 0, \tag{95}$$

agreeing with (53) if

$$\kappa^2 = c^2\rho/\mathrm{T}-b^2. \tag{96}$$

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* An observation during the exceptionally hot weather of last summer recalled my attention to this subject. A train passing at high speed at a distance of not more than 150 yards was almost inaudible. The wheels were in full view, but the situation was such that the line of vision passed for most of its length pretty close to the highly heated ground. It seemed clear that the sound rays which should have reached the observers were deflected upwards over their heads, which were left in a kind of shadow.

The waves originally move towards the less dense parts, and total reflection will ensue when a place is reached, at and after which κ^2 is negative. The case which best lends itself to analytical treatment is when ρ is a linear function of x. κ^2 is then also a linear function; and, by suitable choice of the origin and scale of x, (95) takes the form

$$\frac{d^2\phi}{dx^2}+\tfrac{1}{3}x\phi = 0. \tag{97}$$

The waves are now supposed to come from the positive side and are totally reflected at $x = 0$. The coefficient and sign of x are chosen so as to suit the formulæ about to be quoted.

The solution of (97), appropriate to the present problem, is exactly the integral investigated by Airy to express the intensity of light in the neighbourhood of a caustic.[*] The line $x = 0$ is, in fact, a caustic in the optical sense, being touched by all the *rays*. Airy's integral is

$$\mathrm{W} = \int_0^\infty \cos \tfrac{1}{2}\pi (w^3 - mw)\, dw. \tag{98}$$

It was shown by Stokes[†] to satisfy (97), if

$$x \text{ (in his notation } n) = (\tfrac{1}{2}\pi)^{2/3} m. \tag{99}$$

Calculating by quadratures and from series proceeding by ascending powers of m, Airy tabulated W for values of m lying between $m = \pm 5{\cdot}6$. For larger numerical values of m another method is necessary, for which Stokes gave the necessary formulæ. Writing

$$\phi^{\ddagger} = 2(\tfrac{1}{3}x)^{3/2} = \pi(\tfrac{1}{3}m)^{3/2}, \tag{100}$$

where the numerical values of m and x are supposed to be taken when these quantities are negative, he found when m is positive

$$\mathrm{W} = 2^{\frac{1}{2}}(3m)^{-\frac{1}{4}}\{\mathrm{R}\cos(\phi-\tfrac{1}{4}\pi)+\mathrm{S}\sin(\phi-\tfrac{1}{4}\pi)\}, \tag{101}$$

where
$$\mathrm{R} = 1-\frac{1.5.7.11}{1.2\,(72\phi)^2}+\frac{1.5.7.11.13.17.19.23}{1.2.3.4\,(72\phi)^4}-\ldots, \tag{102}$$

$$\mathrm{S} = \frac{1.5}{1.72\phi}-\frac{1.5.7.11.13.17}{1.2.3\,(72\phi)^3}+\ldots. \tag{103}$$

When m is negative, so that W is the integral expressed by writing $-m$ for m in (98),

$$\mathrm{W} = 2^{-\frac{1}{2}}(3m)^{-\frac{1}{4}}e^{-\phi}\left\{1-\frac{1.5}{1.72\phi}+\frac{1.5.7.11}{1.2\,(72\phi)^3}-\ldots\right\}. \tag{104}$$

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[*] 'Camb. Phil. Trans.,' 1838, vol. 6, p. 379; 1849, vol. 8, p. 595.

[†] 'Camb. Phil. Trans.,' 1850, vol. 9; 'Math. and Phys. Papers,' vol. 2, p. 328.

[‡] Here used in another sense.

The first form (101) is evidently fluctuating. The roots of $W = 0$ are given by

$$\phi/\pi = i - 0{\cdot}25 + \frac{0{\cdot}028145}{4i-1} - \frac{0{\cdot}026510}{(4i-1)^3} + \ldots, \tag{105}$$

i being a positive integer, so that for $i = 2, 3, 4$, etc., we get

$$m = 4{\cdot}3631,\ 5{\cdot}8922,\ 7{\cdot}2436,\ 8{\cdot}4788, \text{ etc.}$$

For $i = 1$, Airy's calculation gave $m = 2{\cdot}4955$.

The complete solution of (97) in series of ascending powers of x is to be obtained in the usual way, and the arbitrary constants are readily determined by comparison with (98). Lommel* showed that these series are expressible by means of the Bessel's functions $J_{\frac{1}{3}}$, $J_{-\frac{1}{3}}$. The connection between the complete solutions of (97), as expressed by ascending or by descending semi-convergent series, is investigated in a second memoir by Stokes.† A reproduction of the most important part of Airy's table will be found in Mascart's 'Optics' (vol. 1, p. 397).

As total reflection requires, the waves in our problem are stationary as regards x. The realised solution of (95) may be written

$$\phi = W \cos(ct + by), \tag{106}$$

W being the function of x already discussed. On the negative side, when x numerically exceeds a moderate value, the disturbance becomes insensible.

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* 'Studien über die Bessel'schen Functionen,' Leipzig, 1868.

† 'Camb. Phil. Trans.,' 1857, vol. 10, p. 106.